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Linear Amplifier and Quasiprobability Distribution Functions for the Squeezed Displaced Fock States

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The nondiagonal P-function (complex and positive) for the output field with the squeezed coherent state as an initial state to a linear amplifier is obtained. Moments of the field operators are calculated. The Wigner distribution function for the output of a linear amplifier is discussed. The input light field of the linear amplifier is assumed to be a squeezed displaced Fock state. Investigation of the Wigner distribution functions at the output fields is carried out, where it is found that the distribution spreads out as it rotates in the phase space with time development. For the case of a squeezed displaced Fock state input it is found that the distribution disintegrates into two subdistributions. The phase distribution for the output light is discussed through the phase function associated with the Wigner function for different inputs.

1. INTRODUCTION

There have been vigorous investigations of the linear interaction of atoms with an optical field (Matsuo, 1993; Hillery and Yu, 1992; Schleich *et al.*, 1992; Agarwal, 1987; Vaccaro and Pegg, 1994; Kim, 1995; Mollow and Glauber, 1967; Carusotto, 1975; Rockower, *et al.*, 1978). The linear light amplifier, which consists of a large number of atoms, amplifies incoming light fields and induces the spontaneous emission of photons. The quantum statistical properties of the output, which should depend on the quantum statistical properties of the input, have been investigated for coherent state input (Mollow and Glauber, 1967; Carusotto, 1975; Rockower *et al.*, 1978) and for squeezed coherent input (Matsuo, 1993; Hillery and Yu, 1992).

The quasiprobability distribution function is a c-number function, not necessarily positive, that allows one to calculate the expectation values of a

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quantum system (Wigner, 1932; Hillery *et al.*, 1984). Recently the quasiprobability functions have become accessible to experimental measurement (Leonhardt and Paul, 1993, 1995) by means of the optical homodyne technique. These measurement schemes have revealed a new facet of the s-parametrized quasiprobability functions in the coherent state basis, namely that the s-parametrized quasiprobability distribution function with fractional values of s ($|s| < 1$) is what is actually seen by the detectors. The value of the parameter s as revealed by these experiments is directly related to the detector efficiency and the amplification of the laser amplifier used in these schemes (Leonhardt and Paul, 1993, 1995). In this language we look for an evolution equation for the quasiprobability which in many cases bears a close anology to the Fokker–Planck equation of classical statistics. A representation of the density matrix with continuous indices is obtained by expanding on the basis of either coherent (Glauber, 1963) or the squeezed states (Yuen, 1976; Kim *et al.*, 1989; Wünsche, 1996). The most widely used representation of this kind is the diagonal P-representation (Glauber, 1963). The resulting P-function (Glauber, 1963) is nonanalytic, not positive definite in general, and therefore cannot be interpreted as a probability density. An off-diagonal representation, the R-representation, was also introduced (Glauber, 1963), but only little use has been made of that tool. The off-diagonal P-representation is given by Drummond and Gardiner (1980) and Gilchrist *et al.* (1997) for the coherent state $|\alpha\rangle$ basis and is generalized by Wünsche (1996) and Obada and Abd Al-Kader (n.d.) for the squeezed state $|z, \alpha\rangle$. In the latter case the density operator ρ has the form

$$\rho = \iint P(\alpha, \beta, z)\Lambda(\alpha, \beta, z)\, d\mu(\alpha, \beta) \tag{1.1}$$

where $P(\alpha, \beta, z)$ is analogous to the P-function, and

$$\Lambda(\alpha, \beta, z) = \frac{|z, \alpha\rangle\langle z, \beta^*|}{\langle z, \beta^*|z, \alpha\rangle} \tag{1.2}$$

and $d\mu(\alpha, \beta) = d^2\alpha\, d^2\beta$ for the positive P-representation and $d\mu(\alpha, \beta) = d\alpha\, d\beta$ for the complex P-representation.

The positive P-representation has the form (Obada and Abd Al-Kader, n.d.)

$$P(\alpha, \beta, z) = \frac{1}{4\pi^2} \exp\left(-\frac{1}{4}|\alpha - \beta^*|^2\right) \times \left\langle z, \frac{1}{2}(\alpha + \beta^*)|\rho|z, \frac{1}{2}(\alpha + \beta^*)\right\rangle \tag{1.3}$$

The c-number function $P(\alpha, \beta, z)$ represents the statistical properties of the density operator and is obviously real, positive, and normalized due to the normalization of the density operator:

$$Tr(\rho(0)) = 1 = \int P(\alpha, \beta, z)\, d^2\alpha\, d^2\beta \qquad (1.4)$$

The eigenvector of the creation operator a^+ is constructed by contour integration from the δ-function (Fan and Xiao, 1996). This motivated us to study the complex P-representation with squeezed state basis. This representation may find application to the excited two-photon coherent state of the radiation field (Xin *et al.*, 1996).

To find the phase distribution of a quantum state is a nontrivial task. The reason for this is that Hermitian phase operators are rare (Carruthers and Nieto, 1968; Barnett and Pegg, 1986). However, one approach that is free of any such problems immediately offers itself: Express the Wigner distribution function (WDF) of this state, which is ordinarily given in the (dimensionless) variables coordinate x and momentum p, in polar coordinates, radius r and angle θ, and integrate over the radius (Tanas *et al.*, 1992; Herzog *et al.*, 1993; Leonhardt and Jex, 1994; also see Special Issue, *Physica Scripta*, **T48**, 1993). The resulting distribution $P^{W}(\theta)$ is periodic in the "phase angle" θ and, for various examples of states, it satisfies all properties required by a proper phase distribution.

The purpose of this article is to examine the statistical properties and phase distribution in full generality, by using the squeezed displaced Fock (SDF) state as an initial input field for the linear amplifier. The use of such nonclassical states not only leads us to a deeper understanding of the nature of light, but also is applicable to the detection of weak signals and quantum communications. We shall consider this in terms of the WDF (Wigner, 1932; Hillery *et al.*, 1984). Due to the properties of the WDF, it always exists, though it may become negative, and has remarkably simple transformation properties.

In Section 2 we briefly discuss the linear light amplifier and introduce the Fokker–Planck equation and its steady-state complex P-function solution in the squeezed-state basis. We also discuss the positive P-function for the linear amplifier. In Section 3 we calculate the characteristic function (CF) for a linear amplifier with SDF state input field and list several field moments. In Section 4 we obtain the Wigner distribution on a linear amplifier with SDF state as initial input field and we discuss important special cases. The behavior of the WDF in three-dimensional diagrams is demonstrated as a function of the interaction time in this section. In Section 5 we introduce the phase distribution function by integrating the WDF over the radial variable, and we also plot the Wigner phase distribution in two cases: squeezed and displaced Fock (number) states as input fields. Conclusions are drawn in Section 6.

2. LINEAR AMPLIFIER

We briefly discuss an optical linear amplifier and its dynamics. We assume that there exist N_T two-level atoms concentrated in a very small region of the space compared with the radiation wavelength, and that a single mode of the electric field interacts with their dipole moments through the atomic transitions. The field frequency is resonant with the atomic transition frequency and the position-dependent variable of the field is eliminated. Suppose that N_1 of the atoms are in the lower state and N_2 in the upper state ($N_T = N_1 + N_2$). The system behaves as an amplifier if $N_1 < N_2$, and as a field attenuator when $N_1 > N_2$. The density operator ρ of the field obeys the following differential equation (Carusotto, 1975):

$$\frac{\partial \rho}{\partial t} = \eta N_2 (2a^+ \rho a - aa^+ \rho - \rho a a^+) + \eta N_1 (2a\rho a^+ - a^+ a \rho - \rho a^+ a) \tag{2.1}$$

where a and a^+ are the usual single-mode photon annihilation and creation operators and η denotes the coupling constant between the atoms and the field. The equation of motion (2.1) can be converted to the Fokker–Planck equation with the appropriate initial field states (Obada and Abd Al-Kader, n.d.; Louisell, 1973; Walls and Milburn, 1994). The statistical properties of the linear amplifier are expressed in terms of the quasidistribution related to the generalized P-function. In this section we are interested in the nondiagonal P-representation with squeezed-state basis. From our results, however, the corresponding formulas for the coherent-state basis can be derived in an obvious way. Moreover, the Glauber (diagonal) P-function and Q-function can be derived from the complex and positive P-repesentations, respectively. We shall review the essential properties of the chosen states for the basis, i.e., squeezed states.

2.1. Squeezed States

The squeezed state $|z, \beta\rangle$ can be found in Yuen (1976) and Kim *et al.* (1989), from which we take a few necessary definitions and results. It is defined by

$$|z, \beta\rangle = D(\beta)S(z)|0\rangle = S(z)D(\beta_0)|0\rangle = S(z)|\beta_0\rangle \tag{2.2}$$

with $\beta_0 = \mu\beta + \nu\beta^*$, where the displacement operator $D(\alpha)$ and squeeze operator $S(z)$ are given by (Yuen, 1976; Kim *et al.*, 1989)

$$D(\alpha) = \exp(\alpha a^{+} - \alpha^{*} a) \qquad \text{and} \qquad S(z) = \exp\left(\frac{z^{*}}{2} a^{2} - \frac{z}{2} a^{+2}\right) \tag{2.3}$$

The operators $D(\alpha)$, $S(z)$ satisfy the transformations

$$D(\alpha) a D^{+}(\alpha) = a - \alpha = A(\alpha), \qquad D(\alpha) a^{+} D^{+}(\alpha) = a^{+} - \alpha^{*} = A^{+}(\alpha) \tag{2.4}$$

$$S(z) a S^{+}(z) = \mu a + \nu a^{+} = b, \qquad S(z) a^{+} S^{+}(z) = \mu^{*} a^{+} + \nu^{*} a = b^{+} \tag{2.5}$$

where z and z^{*} are related to μ and ν by

$$\mu = \cosh|z|, \qquad \nu = \exp(i\phi) \sinh|z|, \qquad z = |z| \exp(i\phi) \tag{2.6}$$

By applying the Bogoliubov transformation (2.5) and using some operator identities (Drummond and Gardiner, 1980 Gilchrist *et al.*, 1997; Obada and Abd Al-Kader, n.d.; Walls and Milburn, 1994), one finds that the projection operator (1.2) satisfies the operator identities

$$a\Lambda(\alpha, \beta, z) = \left\{\mu^{*}\alpha_0 - \nu\left(\beta_0 + \frac{\partial}{\partial\alpha_0}\right)\right\}\Lambda(\alpha, \beta, z) \tag{2.7a}$$

$$a^{+}\Lambda(\alpha, \beta, z) = \left\{\mu\left(\beta_0 + \frac{\partial}{\partial\alpha_0}\right) - \nu^{*}\alpha_0\right\}\Lambda(\alpha, \beta, z) \tag{2.7b}$$

$$\Lambda(\alpha, \beta, z)a = \left\{\mu^{*}\left(\alpha_0 + \frac{\partial}{\partial\beta_0}\right) - \nu\beta_0\right\}\Lambda(\alpha, \beta, z) \tag{2.7c}$$

$$\Lambda(\alpha, \beta, z)a^{+} = \left\{\mu\beta_0 - \nu^{*}\left(\alpha_0 + \frac{\partial}{\partial\beta_0}\right)\right\}\Lambda(\alpha, \beta, z) \tag{2.7d}$$

with the relation between β_0 and β given after (2.2) and a similar relation $\alpha_0 = \mu\alpha + \nu\alpha^{*}$.

The generalized Fokker–Planck equation may be obtained (Drummond and Gardiner, 1980; Gilchrist *et al.*, 1997) in the standard way for the complex P-representation in the case of the linear amplifier. By using equations (2.7) and (1.1) in (2.1), we get

$$\frac{\partial P_c(\alpha, \beta, z, t)}{\partial t} = \left\{\frac{\partial^2}{\partial\alpha_0^2} A + \frac{\partial^2}{\partial\alpha_0\partial\beta_0} 2h + \frac{\partial^2}{\partial\beta_0^2} B - \frac{\partial}{\partial\alpha_0}\eta(N_2 - N_1)\alpha_0 - \frac{\partial}{\partial\beta_0}\eta(N_2 - N_1)\beta_0\right\} P_c(\alpha, \beta, t) \tag{2.8}$$

where

$$A = \eta\mu\nu(N_2 + N_1), \qquad B = \eta\mu^*\nu^*(N_2 + N_1)$$
$$h = \eta(|\mu|^2 N_2 + |\nu|^2 N_1) \tag{2.9}$$

The steady-state solution for the above equation has the form

$$P_c(\alpha, \beta, z) = A_1 \exp\left[\frac{\eta(N_1 - N_2)}{2(AB - h^2)}(-B\alpha_0^2 + 2h\alpha_0\beta_0 - A\beta_0^2)\right] \tag{2.10}$$

where A_1 is the normalization constant. We note the steady-state solution found in the attenuator case only.

Equation (2.10) gives the complex P-function with squeezed-state basis representation. The result when $\nu = 0$, $\mu = 1$ is the complex P-representation for the coherent-state basis (Walls and Milburn, 1994). When we put $\mu = 1$, $\nu = 0$, and $\alpha = \beta^*$ we have the Glauber–Sudarshan P-function for the steady-state field, in the form

$$P(\alpha, \alpha^*) = \frac{\eta(N_1 - N_2)}{\pi N_2} \exp\left[-\frac{\eta(N_1 - N_2)}{N_2}|\alpha|^2\right] \tag{2.11}$$

The steady-state solution to the Fokker–Planck equation in terms of the diagonal P-representation does exist for some systems. The complex P-representation, on the other hand, may take complex values, so that in no sense can it have any probability distribution interpretation. However, it is useful to give exact results for certain problems and physical observables such as all the single correlation functions (Walls and Milburn, 1994). The nondiagonal P-representations and their associated Fokker–Planck equations have been used successfully to study properties of the linear amplifier. Finally, we note that the Fokker–Planck equation (2.8) can be written in terms of the quadratures of the field, i.e., $\alpha = x + iy$ and $\beta = x - iy$; this corresponds to writing the field annihilation operator b, in the squeezed-state basis, as $b = b_1 + ib_2$. Using the resulting equation, one can calculate the variances and covariance in the two quadratures b_1 and b_2. In this representation it is easier to work with the operators b and b^+ rather than a and a^+.

2.2. Positive P-Representation (PPR) for the Linear Amplifier

In this section we briefly discuss the (PPR) given by (1.3) for the linear amplifier. The solution of (2.1) may be found (Hillery and Yu, 1992; Schleich *et al.*, 1992) for a coherent state $|\acute{\alpha}_0\rangle$ as an initial input. In terms of Glauber–Sudarshan diagonal P-representation (Hillery and Yu, 1992; Glauber, 1963; Walls and Milburn, 1994) it has the form

$$P(\acute{\alpha}, t) = \frac{1}{\pi M(t)} \exp\left[-\frac{|\acute{\alpha} - G^*\acute{\alpha}_0|^2}{M(t)} \right] \tag{2.12}$$

with

$$G(t) = \exp[(2k - 1)N_T\eta t + i\omega t] \tag{2.13}$$

and

$$M(t) = \frac{k}{2k - 1}[|G(t)|^2 - 1] \tag{2.14}$$

where ω is an angular frequency and $k = N_2/N_T$, which may be called an atomic population parameter. If all atoms are in the upper state, then $k = 1$; on the other hand, if $k = 0$, then all atoms are in the lower state. Note that $M(t)$ is the average thermal photon number generated by spontaneous emission processes of the atomic system.

We use this result, (2.12), to calculate the density operator with coherent state basis. Substituting in (1.3), we have

$$P(\alpha, \beta, z, t) = \frac{1}{4\pi^3 M(t)} \exp\left[-\frac{1}{4}|\alpha - \beta^*|^2 \right] \int \left\{ \exp\left[-\frac{1}{M(t)} |\acute{\alpha} - G^*\acute{\alpha}_0|^2 \right] \right\} \times \left| \left\langle \frac{1}{2}(\alpha + \beta^*), z | \acute{\alpha} \right\rangle \right|^2 d^2\acute{\alpha} \tag{2.15}$$

By performing the integration, we have

$$\begin{aligned} P(\alpha, \beta, z, t) = {} & \frac{1}{4\pi^2 \mu\, M(t)\sqrt{K_1}} \\ & \times \exp\left[-\frac{1}{4}|\alpha - \beta^*|^2 - \left|\frac{\Delta}{2}\right|^2 + \frac{\nu^*}{2\mu}\Delta^2 + \frac{\nu}{2\mu}\Delta^{*2} - \frac{|G^*\acute{\alpha}_0|^2}{M(t)} \right] \\ & \times \exp\left\{ \frac{1}{K_1}\left[\left(1 + \frac{1}{M(t)}\right)\left|\frac{1}{\mu}\Delta + \frac{G^*\acute{\alpha}_0}{M(t)}\right|^2 - \frac{\nu^*}{2\mu}\left(\frac{1}{\mu}\Delta + \frac{G^*\acute{\alpha}_0}{M(t)}\right)^2 \right] \right\} \\ & \times \exp\left\{ \frac{1}{K_1}\left[-\frac{\nu}{2\mu}\left(\frac{1}{\mu}\Delta^* + \frac{G\acute{\alpha}_0^*}{M(t)}\right)^2 \right] \right\} \end{aligned} \tag{2.16a}$$

where

$$K_1 = \left(1 + \frac{1}{M(t)}\right)^2 - \frac{|\nu|^2}{|\mu|^2} \tag{2.16b}$$

and

$$\Delta = \mu\left(\frac{\alpha + \beta^*}{2}\right) + \nu\left(\frac{\alpha + \beta^*}{2}\right)^* = \frac{1}{2}(\alpha_0 + \beta_0^*) \qquad (2.16c)$$

Using the nondiagonal PPR, we can write the moments of the normally ordered field operators b and b^+ in the form

$$\langle b^{+^n} b^m \rangle = \int \alpha^m \beta^n P(\alpha, \beta, z, t)\, d^2\alpha\, d^2\beta \qquad (2.17)$$

The statistical properties of the output light in this case may be found by using the PPR. An expression for $\langle b^{+^n} b^m \rangle$ has also been worked out, but it is complicated and not very illuminating, so we omit it here.

When we take $\alpha = \beta^*$ in (2.16), we obtain $[1/N(t)]\, Q(\alpha, z, t)$ for the linear amplifier with squeezed coherent state basis (Wünsche, 1996) and coherent state initial input, with $N(t)$ the normalization constant. The resulting function $P(\alpha, \alpha^*, z, t)$ has the form

$$P(\alpha, \alpha^*, z, t) = \frac{\sqrt{K_2}}{\pi} \exp\left[-\frac{1}{\sqrt{K_2}} \{\gamma_1|\gamma_3|^2 + \gamma_2\gamma_3^{*2} + \gamma_2^*\gamma_3^2\} \right]$$
$$\times \exp\{-\gamma_1|\alpha_0|^2 + \gamma_2\alpha_0^2 + \gamma_2^*\alpha_0^{*2} + \gamma_3\alpha_0 + \gamma_3^*\alpha_0^*\} \qquad (2.18)$$

where

$$\begin{aligned}
\gamma_1 &= 1 - \frac{1}{K_1\mu^2}\left(1 + \frac{1}{M(t)}\right) \\
\gamma_2 &= \frac{\nu^*}{2\mu}\left(1 - \frac{1}{K_1\mu^2}\right) \qquad (2.18a) \\
\gamma_3 &= \frac{G(t)\acute{\alpha}_0^*}{K_1\mu M(t)}\left(1 + \frac{1}{M(t)}\right) - \frac{\nu^* G^*(t)\acute{\alpha}_0}{K_1\mu^2 M(t)} \\
K_2 &= \gamma_1 - 4|\gamma_2|^2
\end{aligned}$$

with K_1, $G(t)$, and $M(t)$ given by (2.16b), (2.13), and (2.14), respectively.

In Fig. 1 we plot the PPR function $P(\alpha, \alpha^*, z, t)$ of (2.18), which represents the output of a linear amplifier driven by the coherent state $|\acute{\alpha}_0\rangle$ as an initial input for $\acute{\alpha}_0 = 1$. The coupling constant between the levels and the field is $\eta = 0.2$, the angular frequency is $\omega = 1$, and the lower and upper atom states satisfy $N_2 - N_1 = 1$; then $|G| = \exp(0.2t)$. The squeeze parameter in the basis state (squeezed coherent state) is $|z| = 1$. The interaction time $t = n\pi/4$, $n = 0, 1, \ldots, 4$, is chosen for illustration. This can be compared

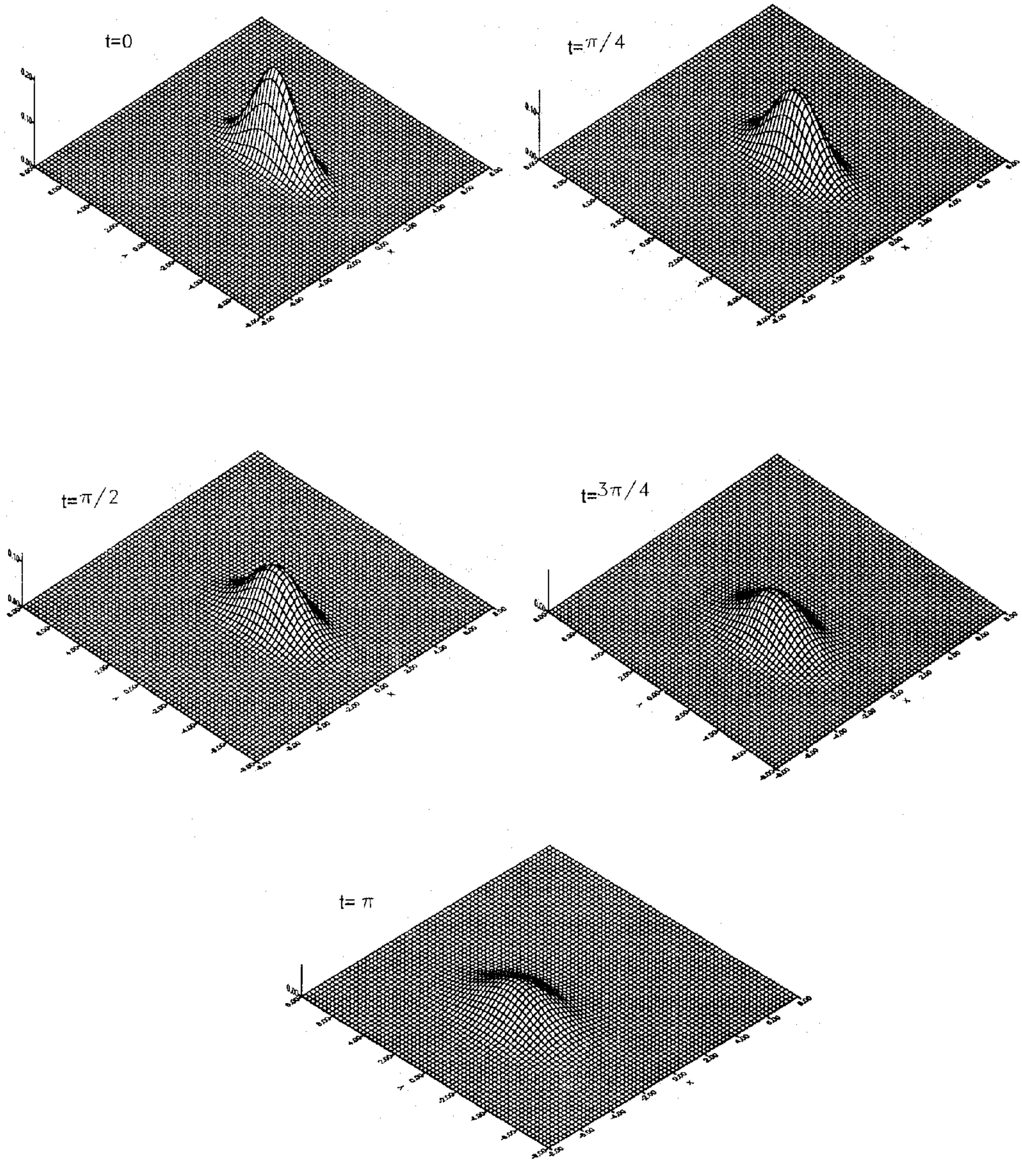


Fig. 1. Temporal behavior of (2.18) for the output of a linear amplifier initially in a coherent state as an input state, with $\acute{\alpha}_0 = 1$. The squeeze parameter for the basis state is $|z| = 1$. The amplifier parameters are $\omega = 1$ and $|G| = \exp(0.2t)$. The interaction time t is shown.

with the case studied in Matsuo (1993). The effect of squeezing in these figures is obvious, in contrast to Matsuo (1993). We notice, however, that as time develops, squeezing diminshes and the PPR spreads out in the phase space. Similar to the coherent basis, we find that the height of the PPR shrinks as the time t increases.

The $P(\alpha, \alpha^*, z, t)$ of (2.18), which defines the Q-function, may be used to formulate the quantum statistical properties of the output light with squeezed-state basis and coherent state as an input field. It is convenient for evaluating antinormal ordered moments, e.g.,

$$\langle b^n b^{+m} \rangle = \int \alpha^n \alpha^{*m} P(\alpha, \alpha^*, z, t)\, d^2\alpha \tag{2.19}$$

where one finds that the antinormal moment has the form

$$\langle b^s b^{+l} \rangle = \sum_{j=0}^{\min(s,l)} \binom{s}{j} \frac{(l)!}{(l-j)!} \left(\frac{\gamma_1}{K_2}\right)^j \left(-\frac{\gamma_2^*}{K_2}\right)^{(s-j)/2} \left(-\frac{\gamma_2}{K_2}\right)^{(l-j)/2}$$
$$\times H_{s-j}\left[\frac{\gamma_1\gamma_3^* + 2\gamma_2^*\gamma_3}{2\sqrt{\gamma_2^* K_2}}\right] H_{l-j}\left[\frac{\gamma_1\gamma_3 + 2\gamma_2\gamma_3^*}{2\sqrt{\gamma_2 K_2}}\right] \tag{2.20}$$

with γ_i, $i = 1, 2, 3$, and K_2 given by (2.18a). The antinormally ordered moments $\langle b^s b^{+l} \rangle$ give sufficient information about the state of the system.

The system described by (2.1) may be solved with the initial state as squeezed displaced Fock (SDF) state, which is our concern in the next sections.

3. THE CHARACTERISTIC FUNCTION (CF) OF A LINEAR AMPLIFIER WITH SDF STATE INPUT FIELD

Cahill and Glauber (1969) defined a general representation function $W(\alpha, s)$ which may be identified with the functions $Q(\alpha)$, $W(\alpha)$, and $P(\alpha)$ when the order parameter s assumes the values -1, 0, and $+1$, respectively. The s-ordered characteristic function $C(\beta, s)$ is defined by

$$C(\beta,s) = Tr[\rho D(\beta, s)] = \exp\left(\frac{s|\beta|^2}{2}\right) C_W(\beta) \tag{3.1}$$

and

$$W(\alpha, s) = \frac{1}{\pi^2} \int C(\beta, s) \exp(\beta^*\alpha - \beta\alpha^*)\, d^2\beta \tag{3.2}$$

The antinormal quasidistribution function $Q(\alpha)$ is discussed for some special cases of the SDF state in Kral (1990a, b), Moller *et al.* (1996), and Man'ko and Wünsche (1997).

Here we introduce the Wigner CF for the linear amplifier with the squeezed displaced Fock states as initial input field by using similar techniques to those used in Matsuo (1993).

The solution of (2.1) in terms of a normally ordered CF is now written as (Matsuo, 1993; Hillery and Yu, 1992; Schleich *et al.*, 1992)

$$C_N^o(\lambda, t) = C_N^{in}[G(t)\lambda]C_N^{th}(\lambda) \tag{3.3}$$

where $C_N^{in}(\lambda)$ is the normally ordered CF for a single-mode input light field for the linear light amplifier, whereas $C_N^{th}(\lambda)$ is the normally ordered single-mode thermal noise CF (Matsuo, 1993), $C_N^o(\lambda, t)$ is the output solution, $G(t)$ is given by (2.13).

We choose the SDF state for an input field for the linear amplifier; these states have been studied extensively (Kral, 1990a, b; Moller *et al.*, 1996; Man'ko and Wünsche, 1997) because of their interesting nonclassical properties and prospective applications in optical communication and interferometery. The SDF state is defined by

$$|\acute{\alpha}_0, z, m\rangle = D(\acute{\alpha}_0)S(z)|m\rangle \tag{3.4}$$

where $S(z)$ and $D(\acute{\alpha}_0)$ are the squeeze and displacement operators given in (2.3); here $|m\rangle$ is the number (Fock) state.

For a squeezed displaced Fock state defined by (3.2),

$$\rho = |\acute{\alpha}_0, z, m\rangle\langle\acute{\alpha}_0, z, m| \tag{3.5}$$

and after minor operator algebra we have the Wigner CF in the form

$$\begin{aligned} C_W^{in}(\lambda) = \exp\Bigg[& -\frac{|\lambda_0|^2}{2} + \lambda_0(\mu\acute{\alpha}_0^* + \nu^*\acute{\alpha}_0) \\ & - \lambda_0^*(\mu\acute{\alpha}_0 + \nu\acute{\alpha}_0^*)\Bigg] L_m(|\lambda_0|^2) \end{aligned} \tag{3.6}$$

where $\lambda_0 = \mu\lambda + \nu\lambda^*$, $\lambda = \mu^*\lambda_0 - \nu\lambda_0^*$, and $L_m^\sigma(x)$ is the associated Laguerre function given by (Gradshteyn and Ryzhik, 1980)

$$L_m^\sigma(x) = \sum_{s=0}^{m} \binom{m+\sigma}{m-s} \frac{(-1)^s}{s!} x^s \tag{3.7}$$

Then the normally ordered CF is given by

$$C_N^{in}(\lambda) = \exp\left[\frac{|\lambda|^2}{2} - \frac{|\lambda_0|^2}{2} + \lambda_0(\mu\acute{\alpha}_0^* + \nu^*\acute{\alpha}_0) - \lambda_0^*(\mu\acute{\alpha}_0 + \nu\acute{\alpha}_0^*)\right] L_m(|\lambda_0|^2) \tag{3.8}$$

The normally ordered CF for the thermal state is written as (Matsuo, 1993; Hillery and Yu, 1992; Schleich *et al.*, 1992)

$$C_N^{th}(\lambda) = \exp[-|\lambda|^2 M(t)] \tag{3.9}$$

Introducing (3.8) and (3.9) in (3.3), we get the output normally ordered CF in the form

$$C_N^o(\lambda, t) = \exp\left\{-[|G(t)|^2|\nu|^2 + M(t)]|\lambda|^2 - \frac{1}{2}\mu\nu^* G^2\lambda^2 - \frac{1}{2}\nu\mu G^{*2}\lambda^{*2} + G\acute{\alpha}_0^*\lambda - G^*\acute{\alpha}_0\lambda^*\right\} \times L_m[|\mu G\lambda + \nu G^*\lambda^*|^2] \tag{3.10}$$

The Wigner CF of the output field is readily written in the form

$$C_W^o(\lambda, t) = \exp\left[-\left(\frac{1}{2}|\lambda|^2\right)\right] C_N^o(\lambda, t) \tag{3.11}$$

We shall be using this CF to calculate the WDF in the next section.

3.1. Moments

From (3.8) we calculate the moments of the photon operators and the field variables at the output of the linear amplifier. The average values of the annihilation and creation operators satisfy the relation

$$\langle a^{+^s} a^l\rangle = \frac{\partial^s}{\partial\lambda^s}\frac{\partial^l}{\partial(-\lambda^*)^l} C_N^o(\lambda, t)|_{\lambda=\lambda^*=0} \tag{3.12}$$

Below we discuss the average values of the normal product of the annihilation and creation operators in some cases.

3.1.1. Squeezed State, i.e., m = 0 in (3.10)

The expectation value (3.12) in this case becomes

$$\langle a^{+s} a^{l} \rangle = \sum_{j=0}^{\min(s,l)} \binom{s}{j} \frac{(l)!}{(l-j)!} (T_2)^j (-T_1)^{(s-j)/2} (-T_1^*)^{(l-j)/2}$$
$$\times H_{s-j}\left[\frac{\alpha_0^*}{\sqrt{2\mu\nu^*}}\right] H_{l-j}\left[\frac{\alpha_0}{\sqrt{2\mu\nu}}\right] \tag{3.13a}$$

with

$$T_1 = -\frac{1}{2}\mu\nu^* G^2, \qquad T_2 = |G|^2|\nu|^2 + M(t) \tag{3.13b}$$

where we have used the Leibniz formula for higher order differentiation and the definition of

$$H_n(x) = \frac{d^n}{dt^n} \exp[2xt - t^2]|_{t=0} \tag{3.14}$$

for the Hermite polynomials (Gradshteyn and Ryzhik, 1980).

A special case of (3.13) giving similar results may be found in Matsuo (1993).

Since the Glauber second-order coherence (correlation) function is defined by (Walls and Milburn, 1994)

$$g^{(2)} = \frac{\langle a^+ a^+ a a\rangle}{\langle a^+ a\rangle^2} \tag{3.15}$$

one can calculate $g^{(2)}$ from (3.13). This is the autocorrelation function of the output light in the case of squeezed state as an initial state (Matsuo, 1993).

3.1.2. Squeezed Displaced Fock State

The expectation values for field operators can be obtained through the appropriate differentiations with respect to λ and λ^*, respectively, as in (3.12). The average values for the first few members are

$$\langle a\rangle = G^*\alpha_0, \qquad \langle a^+\rangle = G\alpha_0^* \tag{3.16}$$

$$\langle a^+ a\rangle = |G|^2\{[(m+1)|\nu|^2 + m|\mu|^2] + |\alpha_0|^2\} + M(t) \tag{3.17}$$

and the normally ordered fourth moment of the photon operators is

$$\langle a^+a^+aa\rangle = m(m-1)\{16|T_1|^2 + 2B_1^2\} + m\{16|T_1|^2 + 4T_1^*T_3^2 + 4T_1T_3^*B_1 + 4T_2B_1 + 4|T_3|^2\} + 2T_2^2 + 4|T_1|^2 + 2T_1^*T_3^2 + 4T_2|T_3|^2 + 2T_1T_3^{*2} + |T_3^2|^2 \tag{3.18}$$

where

$$B_1 = |G|^2(|\mu|^2 + |\nu|^2) \tag{3.19}$$

and T_1 and T_2 are given by (3.13b), while $T_3 = G\acute{\alpha}_0^*$. Substitution of (3.17) and (3.18) into (3.15) yields the coherence function for the output light field for an input light in the SDF state.

4. WIGNER DISTRIBUTION FUNCTIONS OF A LINEAR AMPLIFIER WITH SDF STATE INPUT FIELD

In this section we obtain equations of the WDF for the output of the linear amplifier with coherent-state basis. The WDF is defined as a Fourier transformation of the Wigner CF of (3.2); then

$$W^o_{SDF}(\beta, t) = \frac{1}{\pi^2}\int\left\{\exp\left[-\left(\frac{1}{2} + |G(t)|^2|\nu|^2 + M(t)\right)|\lambda|^2 + \mu\nu^* G^2\lambda^2 + \nu\mu G^{*2}\lambda^{*2}\right] \times \exp[\lambda^*(\beta - G^*\acute{\alpha}_0) - \lambda(\beta^* - G\acute{\alpha}_0^*)]\right\} \times L_m[|\mu G\lambda + \nu G^*\lambda^*|^2]\, d^2\lambda \tag{4.1}$$

After performing the integration, we can write the Wigner function in the form

$$W^o_{SDP}(\beta, t) = \frac{1}{\pi|G|^2\sqrt{k_2}}\exp\left[\frac{1}{k_2}(-\nu_1|\nu_3|^2 + \nu_2\nu_3^{*2} + \nu_2^*\nu_3^2)\right] \times \sum_{s=0}^{m}\sum_{l=0}^{s}\binom{s}{l}\binom{m}{s}\frac{(-1)^s}{(s-l)!}\left(\frac{\nu_1}{k_2}\right)^l\left(\sqrt{-\frac{\nu_2^*}{k_2}}\right)^{s-l}\left(\sqrt{-\frac{\nu_2}{k_2}}\right)^{s-l} \times H_{s-l}\left[\frac{(-\nu_1\nu_3^* + 2\nu_2^*\nu_3)}{2\sqrt{(-k_2\nu_2^*)}}\right]H_{s-l}\left[\frac{(\nu_1\nu_3 - 2\nu_2\nu_3^*)}{2\sqrt{(-k_2\nu_2)}}\right] \tag{4.2}$$

where

$$\nu_1 = [1 + 2M(t)]\,\frac{|\mu|^2 + |\nu|^2}{2|G|^2} - |\nu|^2 \tag{4.3a}$$

$$\nu_2 = [1 - |G|^2 + 2M(t)]\,\frac{\mu\nu^*}{2|G|^2} \tag{4.3b}$$

$$\nu_3 = -\frac{\nu^*}{G^*}(\beta - G^*\acute{\alpha}_0) - \frac{\mu}{G}(\beta^* - G\acute{\alpha}_0^*) \tag{4.3c}$$

and

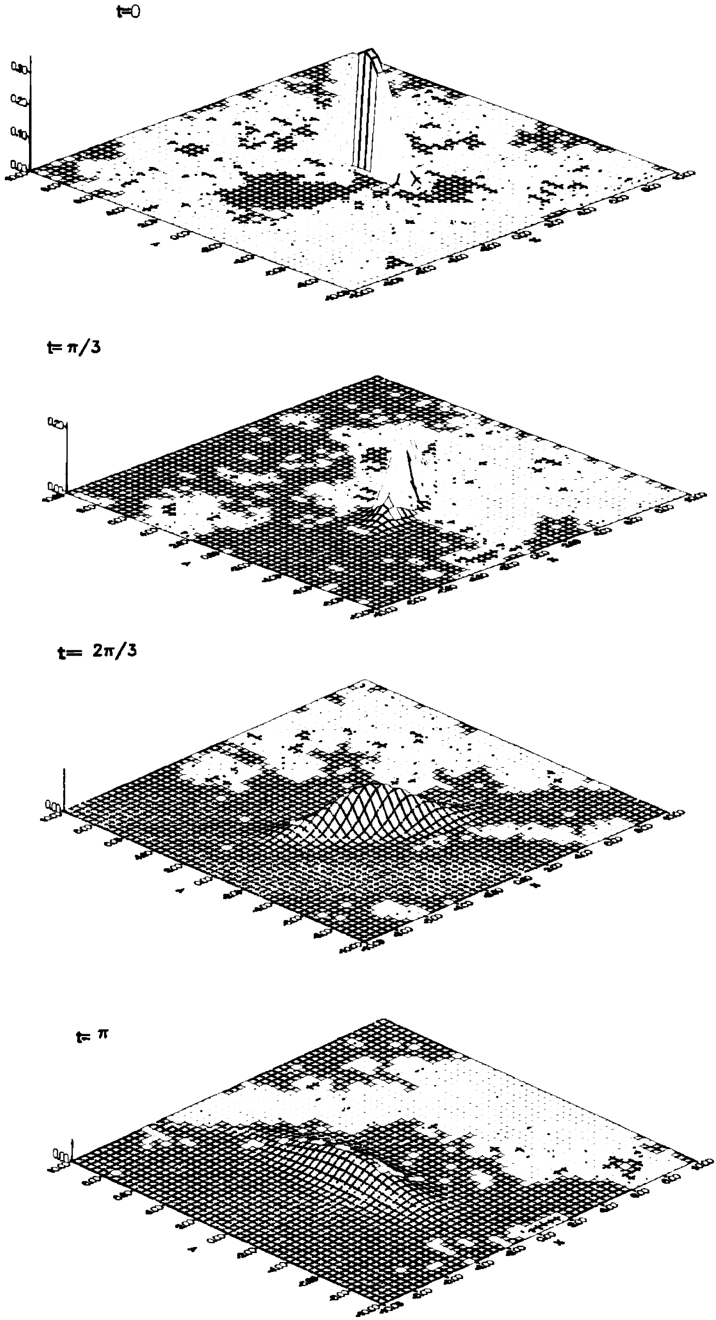
$$k_2 = \nu_1^2 - 4|\nu_2|^2 \tag{4.3d}$$

The formula (4.2) describes the temporal dependence of the WDF for the linear amplifier when the initial input is an SDF state. In what follows we discuss some special cases.

Case 1. Squeezed State Input. When $m = 0$ we get the result of Matsuo (1993) for the input state described by a squeezed state.

In Fig. 2 we demonstrate the three-dimensional time behavior of the WDF for the output of the linear amplifier in the case of a squeezed coherent-state input field. The amplifier and the initial field parameters are given as follows: The displacement parameter $\acute{\alpha}_0$ equals 1. The squeeze parameter z is assumed unity, which implies that $\nu = \sinh(1)$, and $\phi = 0$. The angular frequency is unity, i.e., $\omega = 1$. The atomic population parameter $k = 1$ (i.e., all atoms are in the upper state) and the modified atom-field coupling constant $(2k - 1)N_T\eta$ has the value 0.2; then $|G| = \exp(0.2t)$. We have chosen the interaction time $t = n\pi/3$, $n = 0, 1, 2, 3$, for illustration. By observing the plots, it is apparent that the maximum value of WDF is at position (0, 0) at $t = 0$. With increasing time the maximum value decreases and rotates in a clockwise direction. It can be seen from the figures that the heights of distributions shrink and spread out as the interaction time advances. The spreading of the WDF over the β-plane is shown as time advances. The diminishing of the amount of squeezing as time progresses is exhibited in these figures. In Matsuo (1993) the same figures have been plotted using a different way of computing the WDF (see Fig. 7 in that reference).

Case 2. Displaced Fock State Input. When $\nu = 0$, $\mu = 1$ we obtain the output WDF for the displaced Fock (number) state (Wünsche, 1991; De Oliveira *et al.*, 1990) as an initial state for the input field; in this case the WDF reduces to


t=0
t= π/3
t= 2π/3
t= π

$$W^{o}_{DF}(\beta, t) = \frac{2}{\pi(1 + 2M(t))} \exp\left[-2\frac{|\beta - G^{*}\acute{\alpha}_0|^2}{1 + 2M(t)}\right] \sum_{s=0}^{m} \binom{m}{s} \times \left(\frac{-2|G|^2}{1 + 2M(t)}\right)^{s} L_s 2|\beta - G^{*}\acute{\alpha}_0|^2 \tag{4.4}$$

From (4.4) when $t = 0$ we get

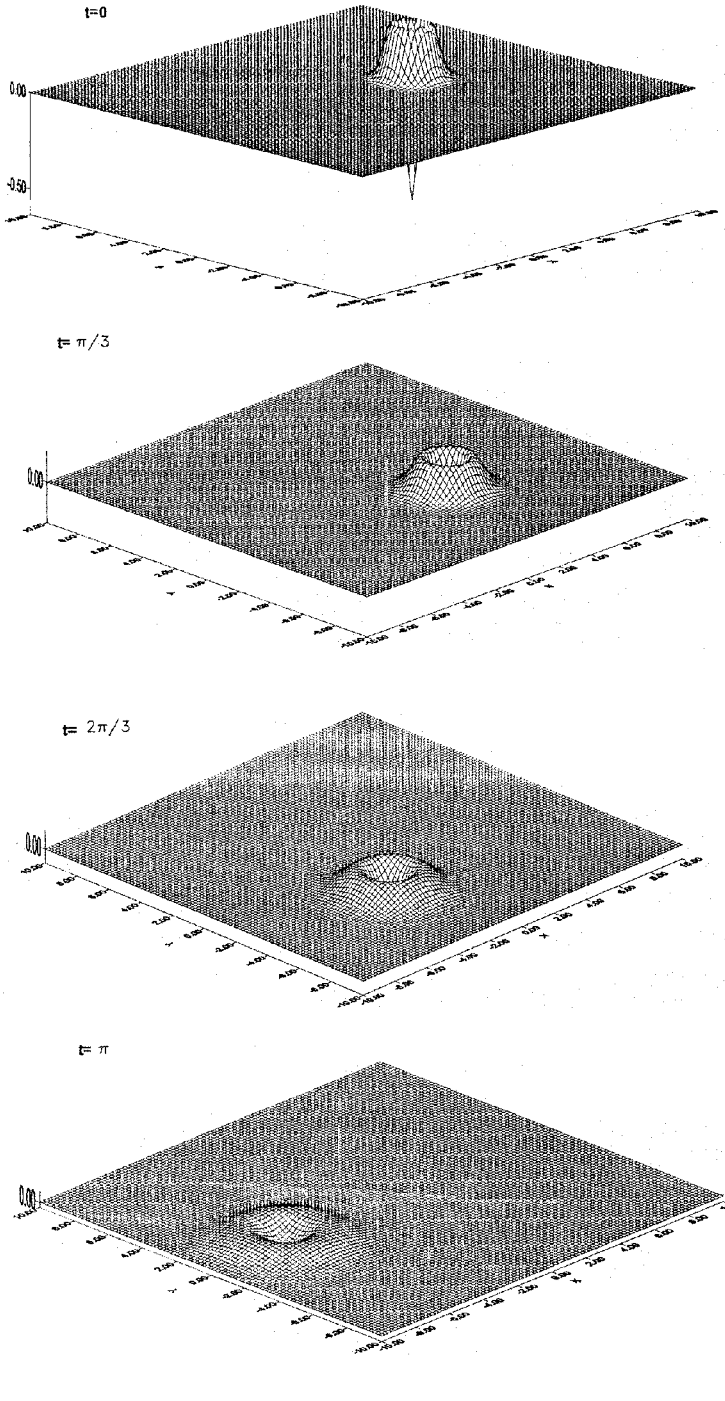
$$W^{o}_{DF}(\beta, 0) = \frac{2}{\pi} \exp[-2|\beta - \acute{\alpha}_0|^2](-1)^m L_m[4|\beta - \acute{\alpha}_0|^2] \tag{4.5}$$

as given in Cahill and Glauber (1969) and De Oliveira *et al.* (1990). We see that there will be oscillations in this function, and negative values occur in some regions.

In Figs. 3 and 4 we demonstrate the three-dimensional time behavior of the WDF for the output of the linear amplifier in the case of a displaced Fock-state input field. We illustrate two different cases: $m = 1$ and $\acute{\alpha}_0 = 3$ in Fig. 3, and $m = 2$ and $\acute{\alpha}_0 = 3$ in Fig. 4. The amplifier and the initial field parameters assume the same values used in Fig. 1. We have chosen the interaction time $t = n\pi/3$, $n = 0, 1, 2, 3$, for illustration. In Fig. 3 when $m = 1$ the WDF has negative values inside the circle $|\beta - G^{*}\acute{\alpha}_0|^2 < [1 + 2M(t)]/2$ centered at $G^{*}\acute{\alpha}_0$, but positive outside this circle. It can be seen from the figures that the heights of the distributions shrink and spread out as the interaction time advances. Note the negative values of the WDF and consequently the nonclassical signature. However, this signature is decreasing with increasing interaction time. The special cases of these figures when $t = 0$ can be found in Tanas *et al.* (1992) and De Oliveira *et al.* (1990).

Finally, in Fig. 5 we demonstrate the temporal behavior of the WDF for the output of the linear amplifier of the squeezed displaced Fock input as given from (4.2). The amplifier parameters have the same values as in Fig. 2 and the initial field parameters are $m = 1$, $\acute{\alpha}_0, = 3$, $\nu = \sinh(1)$, and $\phi = 0$. We have chosen the interaction time $t = n\pi/3$, $n = 0, 1, 2, 3$, for illustration. It is seen clearly that as time develops the distribution breaks up into two subdistributions for this case (see $t = \pi$) and the nonclassical signature is almost lost. The disintegration may be compared with a similar behavior in the Q-function for dissipation in the damped harmonic oscillator as discussed in Kral (1990a, b).

Fig. 2. (*opposite*) Three-dimensional time dependence of the Wigner distribution function for the output of the linear amplifier driven by a squeezed state with $\acute{\alpha}_0 = 1$ and squeeze parameter $|z| = 1$ and direction $\phi = 0$. The amplifier parameters assume the same values as in Fig. 1. Here $X = \mathrm{Re}(\beta)$ and $Y = \mathrm{Im}(\beta)$. The interaction time $t = n\pi/3$, $n = 0, 1, 2, 3$, is chosen for illustration.


t=0
0.00
-0.50
t= π/3
0.00
t= 2π/3
0.00
t= π
0.00

Generally it is seen that, for $m = 0$, the WDF exhibits the standard Gaussian distribution as shown in Matsuo (1993). For $m = 1$ the WDF deviates far away from the Gaussian distribution and becomes negative in some regions of the β-plane. By comparing the plots in Fig. 3 with those in Fig. 5 we can see the effect of the squeeze parameter on the output. We note the spreading of the WDF over the β-plane with increasing time. It is clear that the heights of the distributions shrink and their breadths increase as the interaction time advances. The negativity of the WDF and hence the nonclassical signature decrease with increasing interaction time.

5. PHASE DISTRIBUTION

The phase dependence of quantum noise in squeezed light has provided the motivation for a reanalysis of the phase in quantum optics. Nonclassical light fields are described in terms of quasiprobabilities such as the Wigner or Q (Husimi) functions, and the phase dependence of such distribution functions is a useful parametrization of their properties (Tanas *et al.*, 1992; Herzog *et al.*, 1993; Leonhardt and Jex, 1994; Lynch, 1995; Garraway and Knight, 1992, 1993). It is well known that the Wigner phase distribution is essentially identical to the Pegg–Barnett distribution if the field is dominated by a narrow range of Fock states (Lynch, 1995; De Oliveira *et al.*, 1990; also see Special Issue, *Physica Scripta*, **T48**, 1993). Thus in our approach here we deal with the Wigner phase distribution.

The Wigner phase distribution function is written simply as

$$P^W(\theta) = \int_0^\infty W(|\beta|, \theta)|\beta|\, d|\beta| \tag{5.1}$$

if the Wigner function is expressed in polar coordinates.

We discuss some special cases of the WDF obtained in (4.2).

5.1. Phase Distribution for Squeezed-State Input

When we choose the initial input field to be a squeezed state, i.e., $m = 0$ in (4.2), we have the Wigner distribution function in the form

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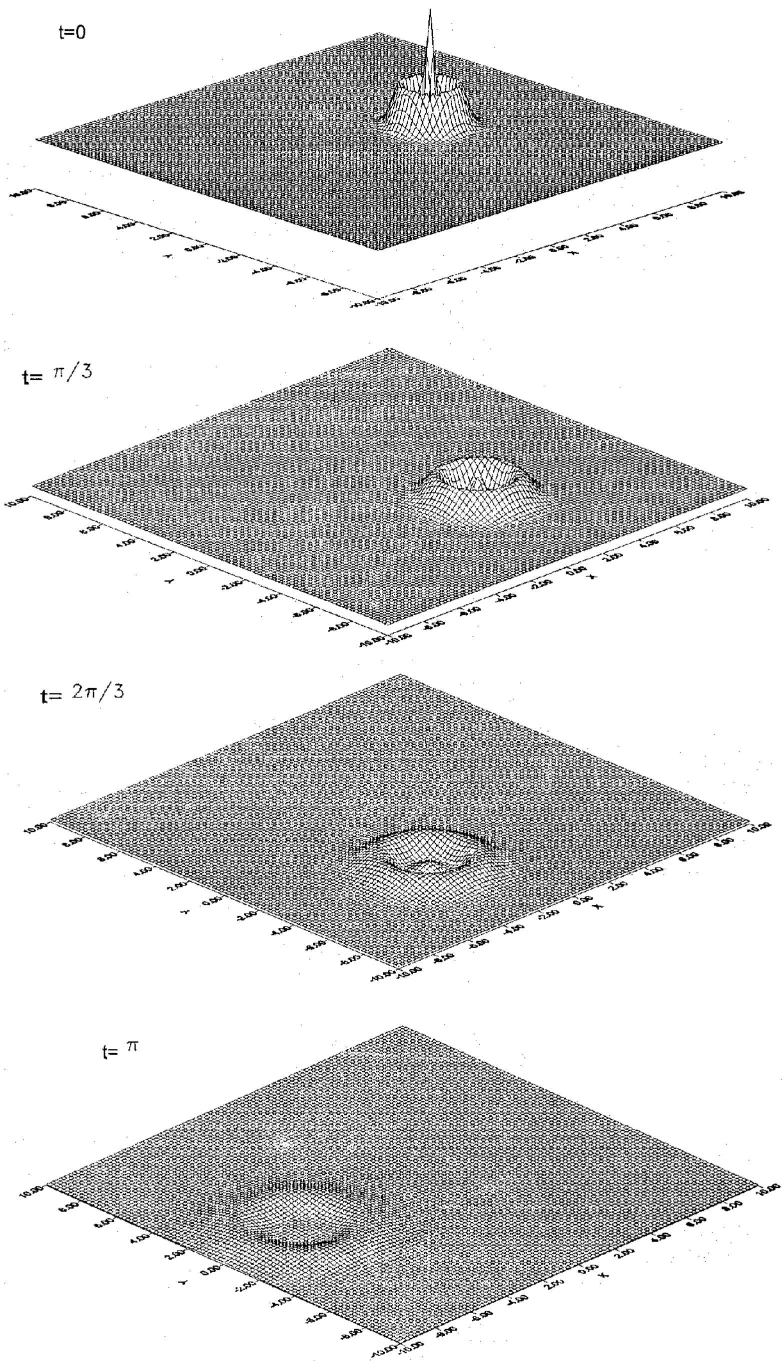
Fig. 3. (*opposite*) Three-dimensional time dependence of the W-function for the output of a linear amplifier driven by a displaced Fock state with $m = 1$ and $\acute{\alpha}_0 = 3$. The amplifier parameters have the same values as in Fig. 1. The interaction time $t = n\pi/3$, $n = 0, 1, 2, 3$, is chosen for illustration. Here $X = \mathrm{Re}(\beta)$ and $Y = \mathrm{Im}(\beta)$.


t=0
t= π/3
t= 2π/3
t= π

$$W^o_{sq}(\beta) = \frac{1}{\pi|G|^2\sqrt{k_2}} \exp\left\{\frac{1}{k_2}\left[A_1|\acute{\alpha}_0|^2 + A_2\acute{\alpha}_0^2 + A_3\acute{\alpha}_0^{*2}\right]\right\}$$
$$\times \exp\left\{\frac{1}{k_2}\left[\frac{A_1}{|G|^2}|\beta|^2 + \frac{A_2}{G_2^*}\beta^2 + \frac{A_3}{G^2}\beta_2^*\right]\right\}$$
$$\times \exp\left\{\frac{1}{k_2}\left[-\left(\frac{A_1\acute{\alpha}_0^*}{G^*} + \frac{2A_2\acute{\alpha}_0}{G^*}\right)\beta - \left(\frac{A_1\acute{\alpha}_0}{G} + \frac{2A_3\acute{\alpha}_0^*}{G}\right)\beta^*\right]\right\} \tag{5.2a}$$

where k_2, ν_1, and ν_2 are given by (4.3), $\beta = |\beta| \exp(i\theta)$, and

$$A_1 = 2\mu\nu\nu_2 + 2\mu^*\nu^*\nu_2^* - \nu_1(|\mu|^2 + |\nu|^2) \tag{5.2a}$$

$$A_2 = \mu^2\nu_2 + \nu_2^*\nu^{*2} - \nu_1\mu\nu^* \tag{5.2b}$$

$$A_3 = \nu^2\nu_2 + \mu^{*2}\nu_2^* - \nu_1\nu\mu^* \tag{5.2c}$$

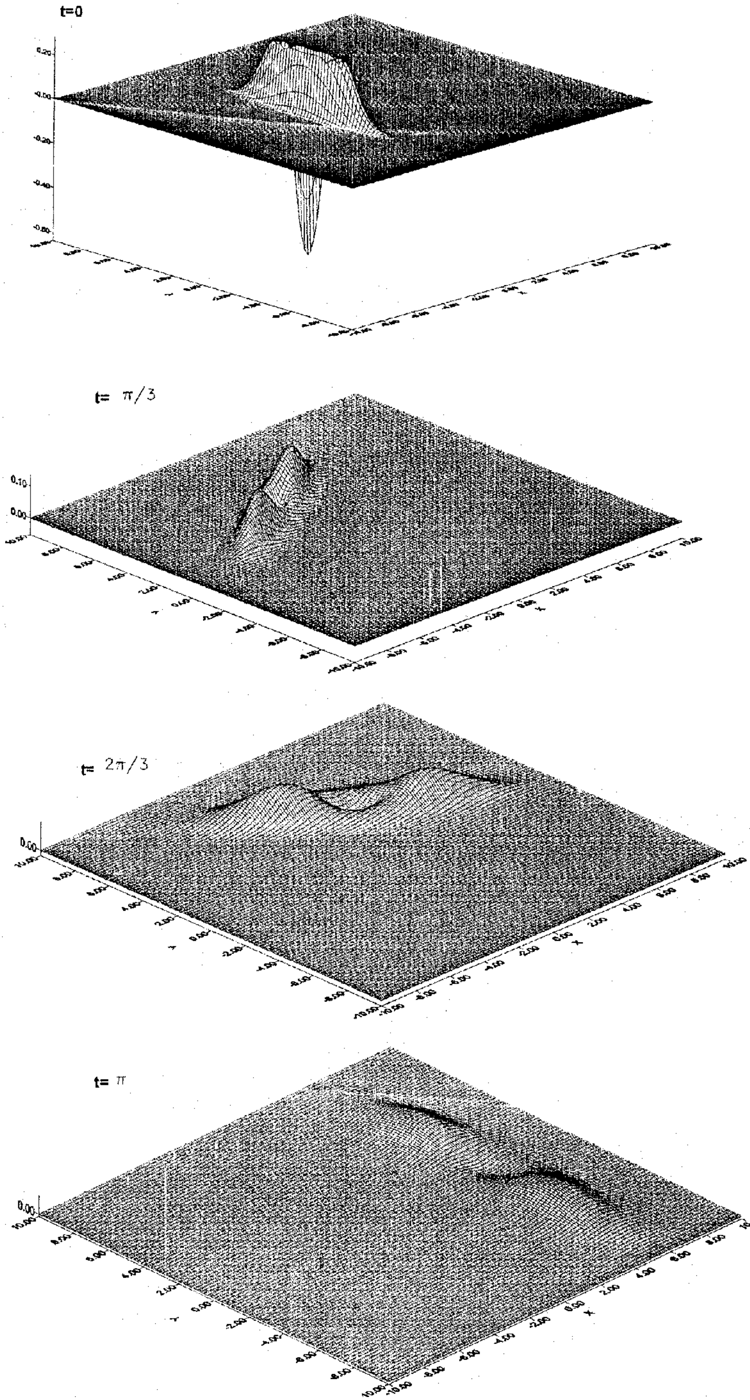
By using (5.2) in (5.1) and using the integral form (Gradshteyn and Ryzhik, 1980)

$$\int_0^\infty x \exp[-ux^2 - 2vx]\, dx$$
$$= \frac{1}{2u} - \frac{v}{u}\sqrt{\frac{\pi}{u}} \exp\left(\frac{v^2}{u}\right)\left[1 - Erf\left(\frac{v}{\sqrt{u}}\right)\right] \tag{5.3}$$

with $[|\arg v| < \pi/2,\ Re\ u > 0]$, we obtain the phase distribution function

$$P^W_{\rm sq}(\theta) = \frac{1}{\pi|G|^2\sqrt{k_2}} \exp\left\{\frac{1}{k_2}[A_1|\acute{\alpha}_0|^2 + A_2\acute{\alpha}_0^2 + A_3\acute{\alpha}_0^{*2}]\right\}$$
$$\times \left\{\frac{1}{2A} - \frac{B}{A}\sqrt{\frac{\pi}{A}} \exp\left(\frac{B^2}{A}\right)\left[1 - Erf\left(\frac{B}{\sqrt{A}}\right)\right]\right\} \tag{5.4}$$

Fig. 4. (*opposite*) Temporal behavior of the W-function for the output of a linear amplifier driven by a displaced Fock state with $m = 2$ and $\acute{\alpha}_0 = 3$. The amplifier parameters have the same values as in Fig. 1. The interaction time $t = n\pi/6$, $n = 0, 1, 2, 3$, is chosen for illustration. Here $X = \text{Re}(\beta)$ and $Y = \text{Im}(\beta)$.


t=0
t= π/3
0.10
0.00
t= 2π/3
0.00
t= π
0.00

where

$$A = \frac{-1}{k_2|G|^2}\{A_1 + A_2 \exp[2i(\theta + \omega t)] + A_3 \exp[-2i(\theta + \omega t)]\} \tag{5.4a}$$

$$B = \frac{1}{2k_2|G|}\{(A_1\alpha_0'^* + 2A_2\alpha_0') \exp[i(\theta + \omega t)]$$
$$+ (A_1\alpha_0' + 2A_3\alpha_0'^*) \exp[-i(\theta + \omega t)] \tag{5.4b}$$

We note that when $t = 0$ we get the phase distribution obtained from the Wigner function of the squeezed state in Tanas *et al.* (1993).

In Fig. 6 we show the phase distribution $P_{sq}^{W}(\theta, t)$ as a function of interaction time t. We take $\alpha_0' = 1$, $\nu = \sinh(1)$, $\phi = 0$, $|G| = \exp(0.2t)$, $k = 1$, $\omega = 1$, and the interaction time $t = n\pi/6$, $n = 0, 1, 2, \ldots, 6$. It is seen that the phase distribution moves and broadens with the increase of time. The bifurcation associated with the squeezed state is noted. A shift of the peak toward $-\pi$ is observed. The plots at $t = 0$ can be found in Tanas *et al.* (1993). The motion of the phase distribution function is related to the rotation in the β-plane of the WDF in Fig. 2.

5.2. Phase Distribution for the Displaced Fock-State Input Light

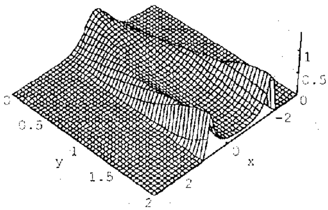
When we choose the initial input as the displaced Fock state, i.e., $\nu = 0$ and $\mu = 1$ in (4.2), we have the WDF for the displaced Fock state in the form

$$W_{DF}^{o}(\beta, t) = \frac{E}{\pi} \exp\{-E|G^*\alpha_0'|^2\} \sum_{s=0}^{m} \sum_{l=0}^{s} \sum_{v,v'=0}^{s-l} \binom{m}{s}\binom{s}{l}$$
$$\times \frac{E^{2s-l}(-1)^{v+v'+l}(|G|)^{v+v'+2s}(|\alpha_0'|)^{v+v'}}{(s-l)!}$$
$$\times \exp[i(v' - v)(\theta - \theta_0 + \omega t)]$$
$$\times |\beta|^{F-2} \exp[-E|\beta|^2 + EJ|\beta|] \tag{5.5a}$$

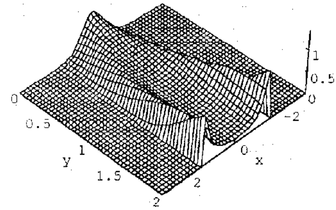
where

Fig. 5. (*opposite*) Temporal behavior of the W-function for the output of a linear amplifier driven by a squeezed displaced Fock state with $m = 1$, $\alpha_0' = 1$, $|z| = 1$, and $\phi = 0$. The amplifier parameters have the same values as in Fig. 1. The interaction time $t = n\pi/6$, $n = 0, 1, 2, 3$, is chosen for illustration. Here $X = \mathrm{Re}(\beta)$ and $Y = \mathrm{Im}(\beta)$.

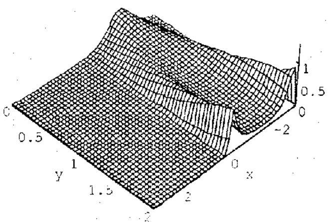
$t = \pi/$



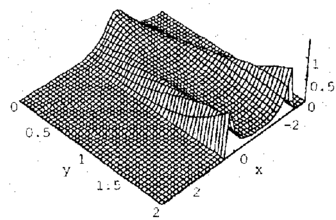
$\mathbf{t = 0}$



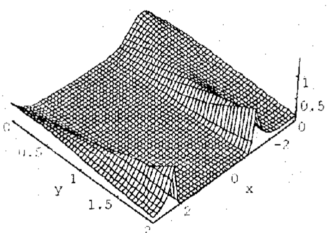
$\mathbf{t = 3\pi/6}$



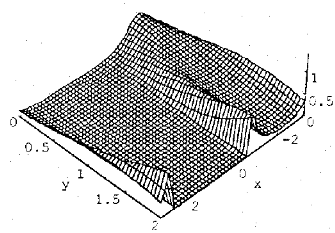
$\mathbf{t = 2\pi/6}$



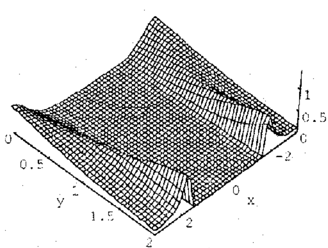
$\mathbf{t = 5\pi/6}$



$\mathbf{t = 4\pi/6}$



$\mathbf{t = 6\pi/6}$



$$E = \frac{2}{1 + 2M(t)}, \qquad F = 2(s - l) - v - \acute{v} + 2$$

$$J = \frac{|\acute{\alpha}_0|\cdot|G|}{2}\cos(\theta - \theta_0 + \omega t)), \qquad \acute{\alpha}_0 = |\acute{\alpha}_0| \exp(\theta_0) \tag{5.5b}$$

By inserting (5.5) in (5.1) and using the integral form

$$\int_0^\infty x^{u-1} \exp(-vx^2 - \gamma x)\, dx$$
$$= (2v)^{-u/2}\Gamma(u) \exp\left[\frac{\gamma}{8v}\right] D_{-u}\left(\frac{\gamma}{\sqrt{2v}}\right) \tag{5.6}$$

where $D_p(x)$ is the parabolic cylinder function (Gradshteyn and Ryzhik, 1980), we obtain the phase distribution as

$$P_{DF}^{W}(\theta, t) = \frac{E}{\pi} \exp\{-E|G^*\acute{\alpha}_0|^2\} \sum_{s=0}^{m} \sum_{l=0}^{s} \sum_{v,\acute{v}=0}^{s-l} \binom{m}{s}\binom{s}{l}$$
$$\times \frac{E^{2s-l}(-1)^{v+\acute{v}+l}(|G|)^{v+\acute{v}+2s}(|\acute{\alpha}_0|)^{v+\acute{v}}}{(s - l)!}$$
$$\times \exp[i(\acute{v} - v)(\theta - \theta_0 + \omega t)]$$
$$\times (2E)^{-F/2}\Gamma(F) \exp\left[\frac{J^2E}{8}\right] D_{-F}\left(\frac{JE}{\sqrt{2E}}\right) \tag{5.7}$$

Equation (5.7) may be written in terms of the error function $Erf(x)$, which is the result in Tanas *et al.* (1993).

Figure 7 shows the dependence of the phase distributions $P_{DF}^{W}(\theta, t)$ of (5.7) associated with the WDF on the interaction time t. We have $m = 1$, $\acute{\alpha}_0 = 1$, $|G| = \exp(0.2t)$, $\omega = 1$. It is clear that the phase graph with two peaks is moving and the peaks become broader with increasing time. The phase information is becoming increasingly lost as time develops. The special case of this plot at $t = 0$ can be found in Tanas *et al.* (1992, 1993). The movement of the $P_{DF}^{W}(\theta, t)$ can be related to the rotation of the WDF in the β-plane as shown in Fig. 3.

Fig. 6. (*opposite*) The phase distribution associated with the Wigner function for a linear amplifier with squeezed state $|\acute{\alpha}_0, z\rangle$ as an initial input field with $\acute{\alpha}_0 = 1$, $|z| = 1$, and $\phi = 0$. The amplifier parameters have the same values as in Fig. 1. Here the phase distribution $P_{\rm sq}^{W}(\theta, t)$ is with $t = n\pi/6$, $n = 0, 1 \ldots, 6$. In this figure X represents θ and Y represents $|z|$.

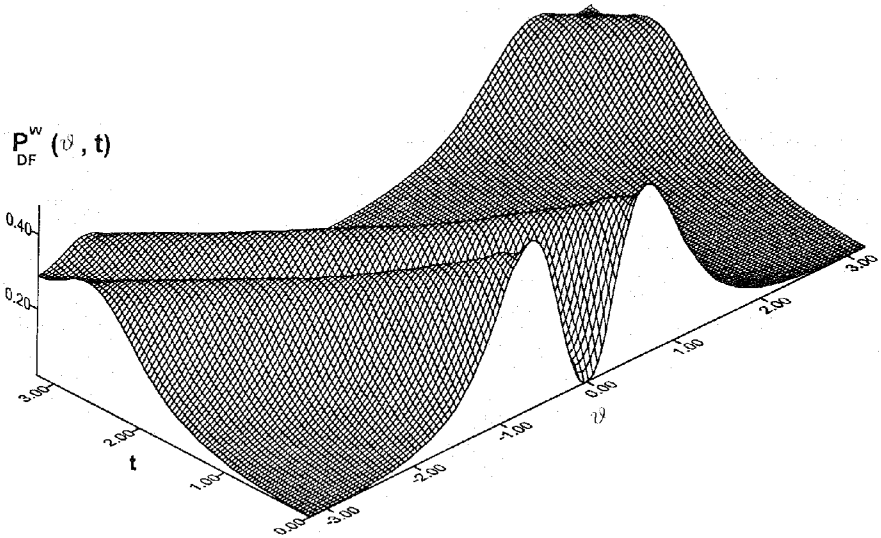


Fig. 7. Temporal behavior of the phase distribution of the Wigner function for a linear amplifier with displaced Fock state $|\acute{\alpha}_0, m\rangle$ as an initial input field with $m = 1$ and $\acute{\alpha}_0 = 1$. The amplifier parameters have the same values as in Fig. 1.

6. CONCLUSIONS

We have studied the Fokker–Planck equation of the linear amplifier master equation using the nondiagonal P-representation with squeezed coherent state basis. We have obtained the steady-state solution of the Fokker–Planck equation using the complex P-representation. From it we obtained the Glauber P-function in the steady state. We have shown that from the time-dependent output P-function in the diagonal P-representation of a linear amplifier we can construct the positive P-representation. The obtained positive P-function characterizes the output of a linear amplifier with squeezed-state basis an initial input for the field in coherent light. We have obtained the Q-function for the output from the positive P-function when $\alpha = \beta^*$. Also, at $t = 0$ the usual Q-function of the squeezed state is retrieved. We have used the squeezed-state basis to obtain the output nondiagonal P-representation for the linear amplifier. The antinormal moments have been calculated. It is hoped that the quasiprobability functions with squeezed-state basis will find applications in quantum measurements.

We have obtained the equation for the WDF of the output field for the linear light amplifier with input squeezed displaced Fock (number) state as an initial state of the field. Some of the inputs have been discussed as special cases, namely squeezed and displaced Fock states. We demonstrated the behavior of the WDF as a function of the interaction time for two special cases. We also showed the behavior of the WDF of the output for the linear amplifier with squeezed displaced Fock (number) state input with time. The resulting plots when $t = 0$ can be compared with the results in De Oliveira *et al.* (1990), Tanas *et al.* (1993), and Lu *et al.* (1989). Our present work was motivated by the desire to realize physically certain specific quantum states (SDF states; Kral, 1990a, b; Moller *et al.*, 1996; Man'ko and Wünsche, 1997) and use them as input for the linear-insensitive amplifier as one of its applications. It is hoped that the SDF states will find application in the quantum nondemolition measurements and quantum optics. They may also find applications in experimental situations that require low noise sensitivity. The physical interpretation of the output of a linear amplifier with SDF states as input differs from that for a squeezed input, as we have shown.

We have discussed the phase distribution produced by integrating the Wigner function over the radius for the special cases above. The results for the phase distributions are illustrated as a function of time. The results here generalize various results reported earlier (Matsuo, 1993; Yuen, 1976; Kim *et al.*, 1989; Wünsche, 1996; De Oliveira *et al.*, 1990; Tanas *et al.*, 1993; Lu *et al.*, 1989).

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